Theorem 8.28

State and prove the Dirichelets test

Statement:

Let $\sum a_n$ be a series of complex terms, whose partial sums from a bounded series

Let $\{b_n\}$ be a decreasing which converge to zero then prove that the series $\sum a_n b_n$ converges.

Proof:

Denote the n^{th} partial sums of the given series $\sum a_n$ then using Abel's partial summation formula, we get

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k) \longrightarrow (1)$$

Hypothesis : 1 The Sequence of partial Sums is bounded

ie) $\{A_n\}_{n=1}^{\infty}$ is bounded

ie) $\exists$ a constant $M > 0$; $\ni$

$$|A_n| \leq M, \quad \forall x \longrightarrow (2)$$

Hyposis : 2 $\{b_n\}$ decreasing zero $n \to \infty$ - for sufficiently large value of x

ie) $b_n \to 0$ as $n \to \infty$

(or)

$$b_{n+1} \to 0 \text{ as } n \to \infty \longrightarrow (3)$$

Consider $|A_n b_{n+1}| \leq |A_n| |b_{n+1}|$

$\leq M |b_{n+1}|$ $\because$ by (2)

$\Rightarrow M \cdot 0 = 0$ as $n \to \infty$ $\because$ by (3)

ie) $\{A_n b_{n+1}\} \to 0$ as $n \to \infty$

ie) The Sequence $\{A_n b_{n+1}\}$ is convergent $\longrightarrow (4)$

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Claim: $\sum_{n=1}^{\infty} a_n b_n$ is convergent

TPT: (e) the sequence of n^{th} partial sums is convergent

(Note that the seq n^{th} partial sums of this series is given by the L.H.S of ①)

$\Rightarrow$ The R.H.S of ① is convergent, when $n \to \infty$

$\Rightarrow$ The sequence $\{A_n b_{n+1}\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty} A_n (b_{n+1} - b_n)$ are convergent.

(of these two $\{A_n b_{n+1}\}$ is already cgt by ④)

$\therefore$ Enough to prove that

The series $\sum_{n=1}^{\infty} A_n (b_{n+1} - b_n)$ is convergent.

Hypothesis: 3 $\{b_n\}$ is decreasing.

let $k < k+1 \Rightarrow b_k \geq b_{k+1}$

$\Rightarrow b_k - b_{k+1} \geq 0$

$|b_k - b_{k+1}| = b_k - b_{k+1} \longrightarrow (*)$

consider

$|A_k (b_{k+1} - b_k)| \leq |A_k| \, |b_{k+1} - b_k|$

$\leq M (b_k - b_{k+1}) \quad \because$ by ⑤ and (*)

$$\sum_{k=1}^{n} A_k (b_{k+1} - b_k) \leq \sum_{k=1}^{n} M (b_k - b_{k+1})$$

$$= M \sum_{k=1}^{n} (b_k - b_{k+1})$$

$$= M (b_1 - b_2 + b_2 - b_3 + \cdots + b_{n-1} - b_n + b_n - b_{n+1})$$

$$= M (b_1 - b_{n+1}) \longrightarrow ⑤$$

Here $\{b_n\}$ is decreasing

$1 \leq n+1 \Longrightarrow b_1 \geq b_{n+1}$

$b_1 - b_{n+1} \geq 0$

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$\therefore$ The Sequence $\{b_1 - b_{n+1}\}_{n=1}^{\infty}$ is sequence of non-negative terms, bounded below by zero

$\Rightarrow$ This sequence is convergent when $n \to \infty$

$\therefore$ The R.H.S of (5) is convergent

ie) $\sum_{n=1}^{\infty} A_n (b_{n+1} - b_n)$ is convergent

Hence the theorem

Theorem: 8.29

(b) Q.P State and prove the Abel's Test.

Statement :

The Series $\sum a_n b_n$ converges if $\sum a_n$ converges and $\{b_n\}_{n=1}^{\infty}$ is a monotonic converges sequence

proof: Let $t_n = a_n b_n$

let $A_n = a_1 + a_2 + \cdots + a_n$ denote the n^{th} partial sums of $\sum a_n$ and

let $V_n = t_1 + t_2 + \cdots + t_n$

$V_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ denote the n^{th} partial sums of $\sum a_n b_n$.

Here,

$A_1 = a_1$

$A_2 = a_1 + a_2$

~~$A_2 = A_1 + a_2$~~

$A_2 - A_1 = a_2$

$A_3 - A_2 = a_3$

$\vdots$

$A_n - A_{n-1} = a_n$

$\therefore V_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

$= A_1 b_1 + (A_2 - A_1) b_2 + (A_3 - A_2) b_3 + \cdots + (A_n - A_{n-1}) b_n$

$= A_1 (b_1 - b_2) + A_2 (b_2 - b_3) + \cdots + A_{n-1} (b_{n-1} - b_n)$

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$$V_n = \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}) + A_n b_n \longrightarrow (1)$$

hypothesis : 1 $\sum a_n$ is convergent

$\Rightarrow$ The sequence of n^{th} partial sums is convergent

$\Rightarrow \{A_n\}$ is convergent

$\Rightarrow \{A_n\}$ is bounded $\longrightarrow (2)$

hypothesis : 2 $\{b_n\}$ is monotonically convergent

$\{b_n\}$ is increasing

$b_k \le b_{k+1}$

$b_k - b_{k+1} \le 0$

$\{b_n\}$ is decreasing

$b_k \ge b_{k+1}$

$b_k - b_{k+1} \ge 0$

In either case $\{b_k - b_{k+1}\}$ is bounded by zero either below (or) above $\longrightarrow (3)$

Thus (2) and (3) $\Rightarrow$ the series,

$\sum A_k (b_k - b_{k+1})$ is convergent as $k \to \infty$

Also (3) $\Rightarrow \{b_n\}$ tends to a finite sum

$\therefore \{A_n b_n\} \to$ a finite sum as $n \to \infty$

In other words, the R.H.S of (1) tends to a finite sum as $n \to \infty$ thus (1) $\Rightarrow \{V_n\}$ is convergent

i.e. $\sum a_n b_n$ is convergent

Hence the theorem.

Rearrangement of Series

Def: 8.21

Let f be a function whose domain is Z^+ and whose range is Z^+, and assume that f is one to one on Z^+. Let $\sum a_n$ and $\sum b_n$ be two series such that

$b_n = a_{f(n)}$ for $n = 1, 2$

Then $\sum b_n$ is said to be a rearrangement of $\sum a_n$.

Theorem: 8.22

Let $\sum a_n$ be an absolutely convergent series having sum S. Then every rearrangement of $\sum a_n$ also converges absolutely and has sum S.

Proof: Let $\sum b_n$ be any arbitrary rearrangement of $\sum a_n$. $\exists$ an onto function (say) $f: Z^+ \longrightarrow Z^+ \ni$ by defn. $\longrightarrow (1)$

$\therefore b_n = a_{f(n)}$

Claim: 1 $\sum b_n$ is an absolute convergent and has the same sum $= S$, where $S = \sum a_n$.

Hypothesis: 1 $\sum a_n$ is absolute convergent

$\therefore \sum a_n$ and $\sum |a_n|$ are convergent

Hypothesis: 2 $\sum a_n$ has the sum $= S$

(ie.) $\sum a_n = \sum |a_n| = S$.

(ie) The sequence of n^{th} partial sums of both $\sum a_n$ and $\sum |a_n|$ converges to S.

To prove: $\sum |b_n|$ is convergent.

(ie) TPT: The sequence of n^{th} partial sums of $\sum |b_n|$ is convergent.

TPT: This Sequence of partial sums of $\sum |b_n|$ is bounded.

Consider. The n^{th} partial sums of $\sum |b_n|$

$= |b_1| + |b_2| + \dots + |b_n|$

$= |a_{f(1)}| + |a_{f(2)}| + \dots + |a_{f(n)}|$ ∴ by ①

$= |a_1| + |a_2| + \dots + |a_n|$

$=$ the n^{th} partial sums of the series $\sum |a_n|$

Hypothesis $\sum |a_n|$ is convergent $\Rightarrow$ the sequence of n^{th} partial sum of $\sum |a_n|$ is convergent

$\rightarrow$ a finite value.

$\Rightarrow$ The sequence of n^{th} partial sum of $\sum |a_n|$ is bdd.

(or) the sequence of n^{th} partial sums of $\sum |b_n|$ is cgt.

$\sum b_n$ is absolute convergent.

Claim: 2 $\sum |b_n|$ (or) $\sum b_n$) has the same sum S

TPT : The sequence of n^{th} partial sums of $\sum |b_n|$ converge to S.

for this purpose let,

$t_n = b_1 + b_2 + \dots + b_n$ denote the n^{th} partial sums $\sum b_n$ and

$s_n = a_1 + a_2 + \dots + a_n$ denote the n^{th} partial sums of $\sum a_n$

Hypothesis : $\sum a_n = S$ ——→ ⊕

The sequence of n^{th} partial sums of $\sum a_n$ converge to S.

ie) The sequence $\{s_n\}$ converges to S

(or) $\lim_{n \to \infty} s_n = S$

By defn of limit of a sequence

a given $\epsilon > 0$, $\exists$ $N \in z^+$ such that

$$|s_n - S| < \epsilon/2 \quad (n \geq N)$$

fix $n = N$

$$|s_N - S| < \epsilon/2 \longrightarrow (3)$$

$$(*) \Rightarrow \quad S = \sum_{n=1}^{\infty} a_n$$

$$\therefore S = a_1 + a_2 + \cdots + a_N + a_{N+1} + \cdots + a_n + \cdots$$

S_N = the sum of N^{th} partial sum.

$$\therefore S_N = a_1 + a_2 + \cdots + a_N.$$

Now consider,

$$|s_n - S| = |S - S_N|$$

$$= |(a_1 + a_2 + \cdots + a_N + a_{N+1} + \cdots + a_n + \cdots) - (a_1 + a_2 + \cdots + a_N)|$$

$$= |a_{N+1} + a_{N+2} + \cdots + a_n + \cdots|$$

$$= \sum_{n=1}^{\infty} |a_{N+k}|$$

We have show that

$$\sum_{k=1}^{\infty} |a_{N+k}| = |s_n - S|$$

$$< \epsilon/2 \longrightarrow (2)$$

Finally, it remains to show that $\{t_n\} \to S$

consider, $|t_n - S|$ sub and add s_N

$$= |t_n - S - S_N + S_N|$$

$$\leq |t_n - S_N| + |s_n - S|$$

$$\leq |t_n - s_n| + \epsilon/2 \longrightarrow (4)$$

Choose a positive integer m, such that

$\{1, 2, \ldots, N\} \subset \{f(1), f(2), \ldots, f(m)\} \longrightarrow (6)$

Let $n > m$

By defn of rearrangement f is onto

$\Rightarrow f(n) > f(m)$

$\Rightarrow f(n) > N$ by (6)

Consider, $|t_n - s_N|$

$= |(b_1 + b_2 + \cdots + b_n) - (a_1 + a_2 + \cdots + a_N)|$

$= |(a_{f(1)} + a_{f(2)} + \cdots + a_{f(n)}) - (a_1 + a_2 + \cdots + a_N)|$

$= |a_{N+1} + a_{N+2} + \cdots + a_n|$

$= \sum_{k=1}^{n} |a_{N+k}|$

$\leq \sum_{k=1}^{\infty} |a_{N+k}|$

$< \epsilon/2$ ∵ by (2) $\longrightarrow$ (5)

Substituting eqn (5) in eqn (4), we get

$|t_n - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$|t_n - S| < \epsilon$

$\Rightarrow \{t_n\} \to S$

Hence $\sum |b_n| = S$.

Hence the theorem.

Thm 8.33 <u>Riemann's Theorem on Conditionally Convergent Series:</u>

Let $\sum a_n$ be a conditionally convergent series with real-valued terms. Let x and y be given numbers in the closed interval $[-\infty, +\infty]$ with $x \leq y$. Then there exists a rearrangement $\sum b_n$ of $\sum a_n$

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Such that,

$$\lim_{n\to\infty} \inf t_n = x \quad \text{and} \quad \lim_{n\to\infty} \sup t_n = y.$$

where $t_n = b_1 + \cdots + b_n$

Proof: Discarding terms of a series which are zero does not affect its convergence or divergence.

Hence we might as well assume that no terms of $\sum a_n$ are zero. i.e.

Let p_n denote the n^{th} positive term of $\sum a_n$ and let q_n denote its n^{th} negative term.

Then $\sum p_n$ and $\sum q_n$ are both divergent series of positive terms.

Next, construct two sequence of real numbers say $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \quad \text{with } x_n < y_n, \; y_1 > 0$$

The Idea of the proof is now quite simple. we take just enough (say k_1) positive terms so that,

$$p_1 + \cdots + p_{k_1} > y_1,$$

followed by just enough (say r_1) negative terms so that

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} < x_1.$$

Next we take just enough further positive terms so that,

$$p_1 + \cdots + p_{k_1} - q_1 - \cdots - q_{r_1} + p_{k+1} + \cdots + p_{k_2} - q_{r_1+1} - \cdots - q_{r_2} < x_2.$$

these steps are possible since $\sum p_n$ and $\sum q_n$ are both divergent series of positive terms. If the process is continued in this way,

we obviously obtain a rearrangement of $\sum a_n$. we leave it to the reader to show that the partial sums of this rearrangement have limit superior y and limit Inferior x.

Double Sequence

Def: A function 'f' whose domain is $Z^+ \times Z^+$ is called a double sequence.

Note: We shall be interested only in real or complex valued double sequences.

Defn: 8.38

If $a \in C$, we write $\lim\limits_{p,q \to \infty} f(p,q) = a$ and we say that the double sequence 'f' converges to a provided that the following condition is satisfied,

For every $\epsilon > 0$, there exists an N such that

$|f(p,q) - a| < \epsilon$ whenever both $p > N$ and $q > N$.

Theorem: 8.39

Assume that $\lim\limits_{p,q \to \infty} f(p,q) = a$. For each fixed p. assume that the limit $\lim\limits_{q \to \infty} f(p,q)$ exists then the limit $\lim\limits_{p \to \infty} \left(\lim\limits_{q \to \infty} f(p,q) \right)$ also exists and has the value a.

Proof: Note: $\lim\limits_{p,q \to \infty} f(p,q)$ is called a 'Double Limit'

$\lim\limits_{p \to \infty} \left(\lim\limits_{q \to \infty} f(p,q) \right)$ is called an Iterated

$\lim\limits_{q \to \infty} \left(\lim\limits_{p \to \infty} f(p,q) \right)$. Limit

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